Coherent Forecasting of Ordinal Time Series

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Introduction

Ordinal time series $X_1, \ldots, X_n$ from process $(X_t)_{\mathbb{Z}}$ with ordered qualitative range $\mathcal{S} = \{s_0, s_1, \ldots, s_d\}$, $s_0 < \ldots < s_d$.

Notations: pmf $\boldsymbol{p} = (p_0, \ldots, p_d)^\top$ with $p_j = P(X = s_j)$, cdf $\boldsymbol{f} = (f_0, \ldots, f_{d-1})^\top$ with $f_j = P(X \leq s_j)$, and $f_d = 1$.

Applications in various fields of practice, such as environmental science, econometrics, finance, or health science, see Liu et al. (2022), Weiß (2020), Weiß & Swidan (2024b).

Typically, range $\mathcal{S}$ **rather small** (or only few categories attained during observation period), such as 3–6 categories.

Major aim of time series analysis:

predict future outcomes of underlying process (DGP).

Recent, very comprehensive review by Petropoulos et al. (2022):

only little research on **discrete**-valued processes, and if so,

then clear focus on count time series (quantitative!).

There, **coherent forecasting** by Freeland & McCabe (2004):

approach for **point forecasts** (PFs) coherent

if generated forecasts only attain values in actual range.

Coherency requirement can also be extended to

prediction intervals (PIs) and **pmf forecasts** (PMFFs).

Comprehensive analyses of coherent PFs, PIs, and PMFFs for *count* time series by Homburg et al. (2019, 2021, 2023).

Present research: comprehensive investigation of coherent forecasting for *ordinal* time series.

Outline:

- Coherent ordinal PFs, PIs, and PMFFs, criteria for evaluating their forecast performance.
- Simulation study: forecast performance for various ordinal DGPs, considering effect of estimated model parameters.
- Application to ordinal t.s. about air quality in Beijing.

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Approaches & Evaluation

Coherent Point Forecasts: h-step-ahead condit. median:

$$\hat{x}_{n+h} \;=\; \min\Big\{y\in\mathcal{S}\;\Big|\;P\big(X_{n+h}\leq y\mid x_n,\ldots,x_1\big)\geq 0.5\Big\}.$$

PF value equiv. as "rank count" $\hat{i}_{n+h}\in\{0,\ldots,d\}$ (Weiß, 2020), defined by $\hat{x}_{n+h}=s_{\hat{i}_{n+h}}$, how many categories apart from s_0.

Remark: Pruscha (1995) and Pruscha & Göttlein (2003) define mean-based PF related to $\hat{i}_{n+h}$, but not coherent.

Performance metrics: $\text{PCP} \;=\; P\big(X_{n+h}=\hat{x}_{n+h}\;\big|\;x_n,\ldots,x_1\big)$.

To use natural ordering of $\mathcal{S}$: "k-nearest-neighbour matching",

$$k\text{-NNM} \;=\; P\Big(X_{n+h}\in\big\{s_{\max\{0,\,\hat{i}_{n+h}-k\}},\ldots,s_{\min\{d,\,\hat{i}_{n+h}+k\}}\big\}\;\Big|\;x_n,\ldots,x_1\Big).$$

Coherent Interval Forecasts for coverage $p_{\text{cov}} \in (0;1)$:

$$P(s_{\text{l}} \leq X_{n+h} \leq s_{\text{u}} \mid x_n, \ldots, x_1) \ \geq \ p_{\text{cov}}.$$

Computational scheme from Homburg et al. (2021) applicable, also their performance criteria based on coverage.

However, rather poor performance in simulation study, because range $\mathcal{S}$ typically rather small, which often causes degenerate or trivial PIs.

$\Rightarrow$ Better kind of "trustworthy" forecasts: PMFFs.

Coherent Pmf Forecasts: PMFFs intensively discussed for count time series, among others McCabe et al. (2011) and Homburg et al. (2023). Pruscha (1995) and Pruscha & Göttlein (2003) used PMFFs also for ordinal t.s.

While PF $\in \mathcal{S}$ and PI $\subseteq \mathcal{S}$ and thus coherent,
PMFF takes full predictive pmf of $X_{n+h} \mid x_n, \ldots, x_1$ as forecast.

PMFF $(d+1)$-dimensional vector of conditional probabilities.
Maximally informative forecast value, also **coherent**
as range $\mathcal{S}$ of ordinal time series is PMFF's support.

PMFF notations:

- If true model for PMFF computation, PMFF vector $\widehat{\boldsymbol{p}}_0^{(n,h)}$ ("true PMFF"), where $\widehat{p}_{0,j}^{(n,h)} = P(X_{n+h} = s_j \mid x_n, \ldots, x_1)$.
- If estimated model,then $\widehat{\boldsymbol{p}}^{(n,h)}$ and $\widehat{p}_j^{(n,h)}$ instead.

Evaluate **forecast inaccuracy** due to estimation uncertainty by MSE-based criteria like in Homburg et al. (2023).

Global MSE: $\left\|\widehat{\boldsymbol{p}}^{(n,h)} - \widehat{\boldsymbol{p}}_0^{(n,h)}\right\|^2 = \sum_{j=0}^{d} (\widehat{p}_j^{(n,h)} - \widehat{p}_{0,j}^{(n,h)})^2.$

Local MSEs:
$$\sum_{j=0}^{d} (\widehat{p}_j^{(n,h)} - \widehat{p}_{0,j}^{(n,h)})^2\, \mathbb{1}(\widehat{f}_{0,j}^{(n,h)} \leq 0.25),$$
$$\sum_{j=0}^{d} (\widehat{p}_j^{(n,h)} - \widehat{p}_{0,j}^{(n,h)})^2\, \mathbb{1}(\widehat{f}_{0,j}^{(n,h)} \geq 0.90).$$

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Performance Analyses

Comprehensive simulation study, where simulated time series $x_1, \ldots, x_n$ used for ML-model fitting and forecast computation.

Various DGPs from

- WDARMA family of Weiß & Swidan (2024a),
- logit AR(1) model of Fokianos & Kedem (2003),
- soft-clipping AR family of Weiß & Swidan (2025),
- logit(μ) HMM of Weiß & Swidan (2024b).

PFs, PIs, and PMFFs for forecast horizons $h = 1, \ldots, 5$, performance evaluation by above metrics.

Detailed simulation results in main paper.

Main findings:

- Estimation uncertainty only little effect on coherent PFs due to discreteness, but reliability of PFs severely depends on actual DGP and parametrization.
- PIs for ordinal t.s. of limited use, as true coverage often far beyond p_{cov}, so PIs for different DGPs hardly comparable.
- Forecast performance of PMFFs not suffer from discreteness, well explainable behavior w.r.t. sample size, model structure, and extent of serial dependence.

PMFFs also practical advantage of providing full information about forecast distribution to user, can judge which category from $\mathcal{S}$ observed in future with which probability.

Final recommendation:

We discourage use of PIs for ordinal processes, but combination of PF and PMFF attractive solution for practice.

PF and PMFF can also be combined visually in well-interpretable way, see subsequent data application.

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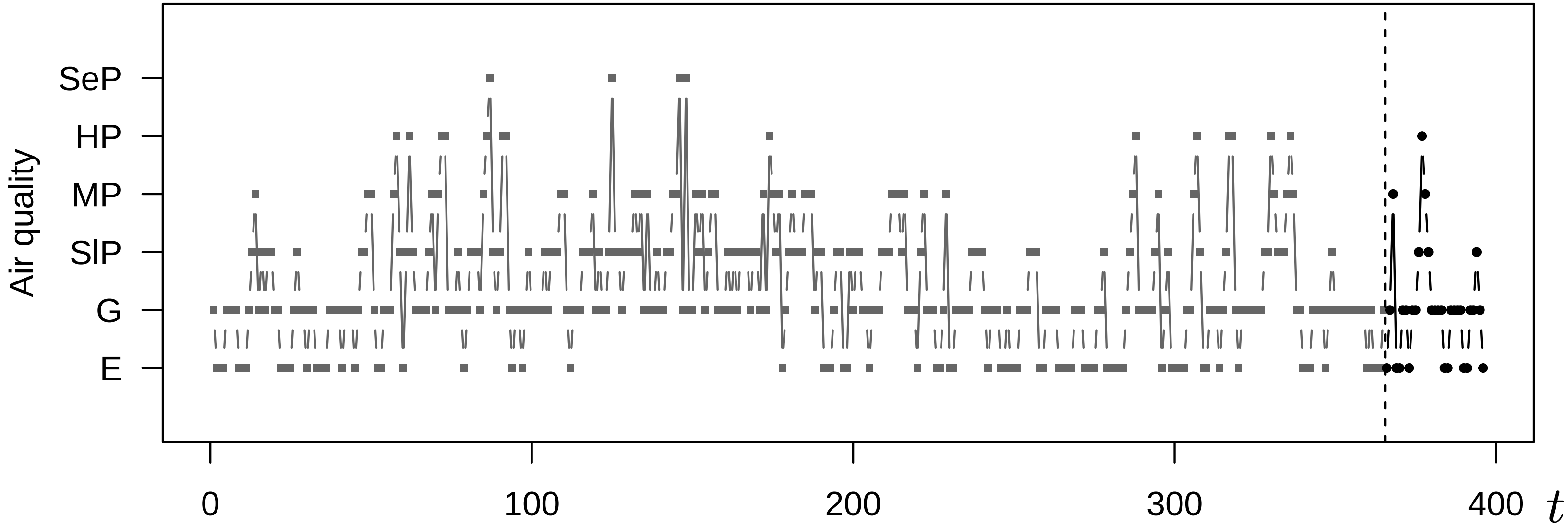
Coherent Forecasting of Air Quality in Beijing

Data Application

Ordinal t.s. on daily air quality in Beijing from Liu et al. (2022), categories $s_0 =$"excellent" (E), ..., $s_5 =$"sev. polluted" (SeP). Data from 2018 for model fitting ($n = 365$), from January 2019 for out-of-sample forecasting.



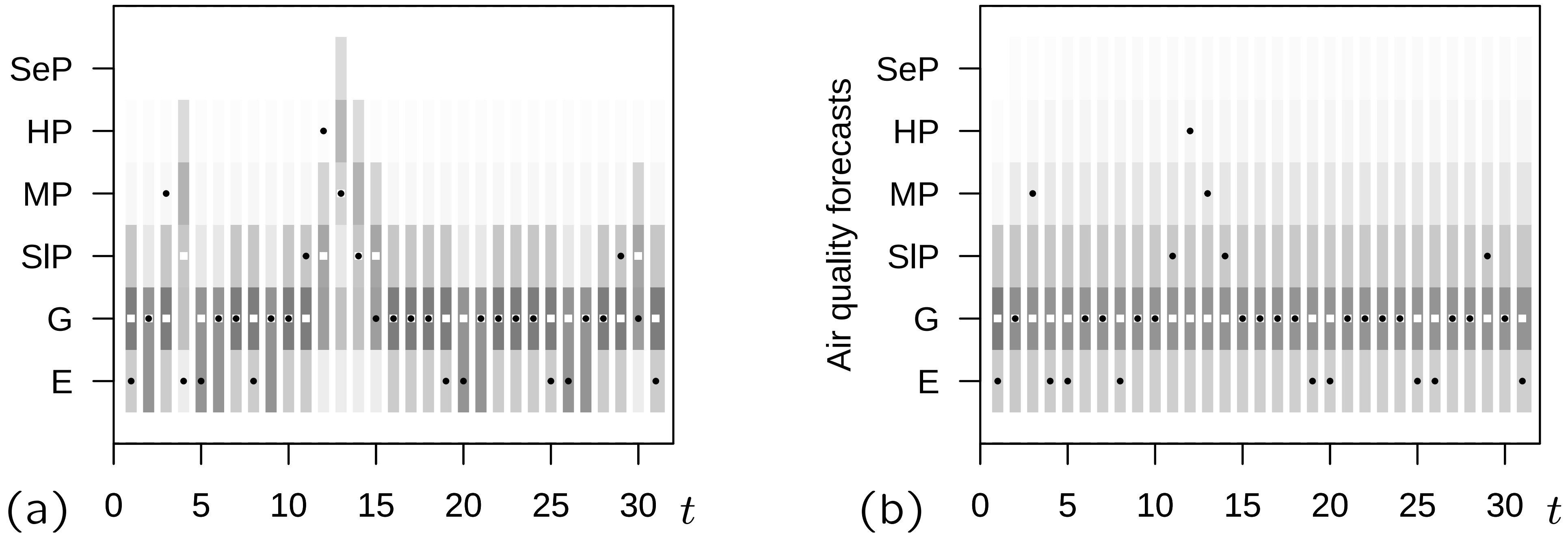
Weiß & Swidan (2024a) used WDAR(1) model for 2018 data.

Data Application: Air Quality in Beijing

PMFFs can be expressed by stripes of gray levels, complemented by PFs as white square.

(a) 1-step-ahead forecasts for $t = 1, \ldots, 31$ given x_{n+t-1},

(b) h-step-ahead forecasts ($h = 1, \ldots, 31$) given $x_n = s_1$.

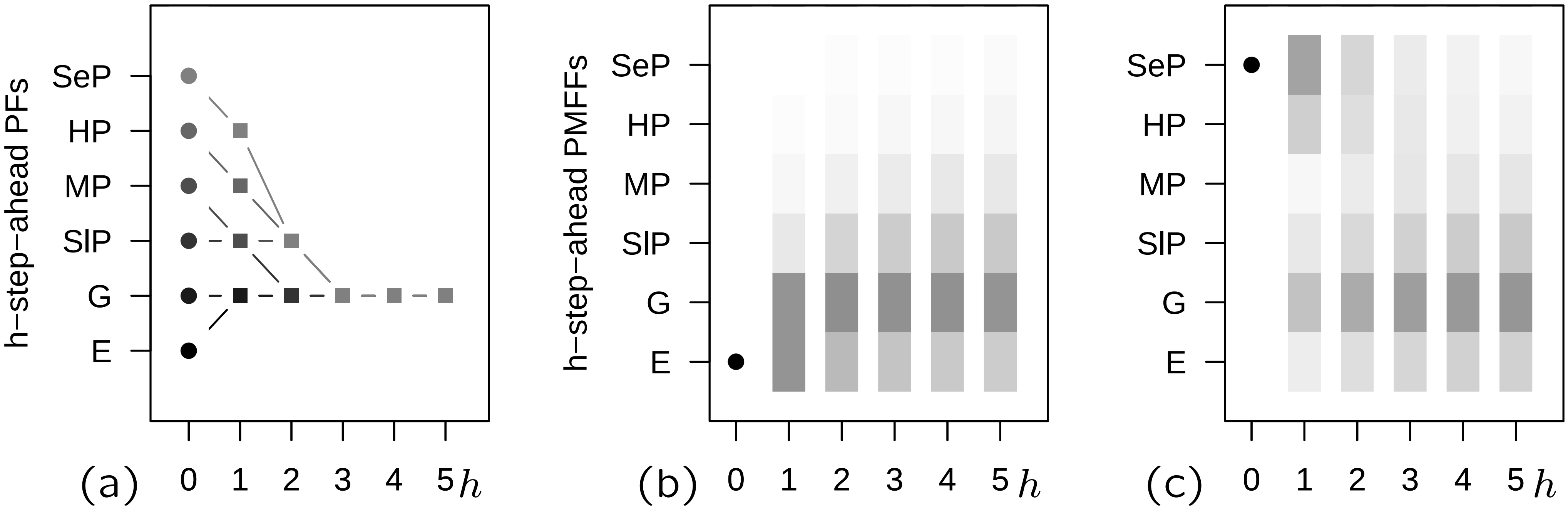


Data Application: Air Quality in Beijing

h-step-ahead forecasting based on fitted WDAR(1) model,

last observation as circle at $h = 0$.

PFs in (a), and PMFFs in (b) and (c).



Data Application: Air Quality in Beijing

Possible application: risk analysis, where

- "$X_{n+t} \geq s_4$" $\approx$ high risk for people's health,
- "$X_{n+t} \leq s_1$" $\approx$ low-risk day.

Using PMFFs, we compute

$$P(\text{“low risk”} \mid x_{n+t-1}) \approx \begin{cases} 0.846 & \text{if } x_{n+t-1} = s_0, \\ 0.714 & \text{if } x_{n+t-1} = s_1, \\ 0.449 & \text{if } x_{n+t-1} = s_2, \\ 0.316 & \text{if } x_{n+t-1} \geq s_3; \end{cases}$$

$$P(\text{“high risk”} \mid x_{n+t-1}) \approx \begin{cases} 0.024 & \text{if } x_{n+t-1} \leq s_2, \\ 0.157 & \text{if } x_{n+t-1} = s_3, \\ 0.421 & \text{if } x_{n+t-1} = s_4, \\ 0.554 & \text{if } x_{n+t-1} = s_5. \end{cases}$$

Other applications: identify "anomalous" observations (low predictive probability), such as days 3, 4, 12 above; impute missing values in ordinal time series.

Conclusions

- Coherent PFs, PIs, and PMFFs for ordinal time series, corresponding metrics for performance evaluation.
- Use of PIs discouraged, but combination of PF and PMFF attractive for practice.
- Visual combination of PF and PMFF well-interpretable, various applications in practice.

Future research:

- accounting for estimation uncertainty via resampling,
- risk forecasting in ordinal time series.

Thank You for Your Interest!

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