Non-Parametric Tests for Spatial Dependence

Christian H. Weiß

Department of Mathematics & Statistics

Helmut Schmidt University, Hamburg

Hee-Young Kim

Department of Bigdata Science

Korea University, Sejong, South Korea

Helmut Schmidt Universität
Universität der Bundeswehr Hamburg

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Ordinal Patterns in Time Series

Introduction

Bandt & Pompe (2002) introduced **ordinal patterns** (OPs) as complexity measures for time series characterized by "*simplicity, extremely fast calculation, robustness, and invariance with respect to nonlinear monotonous transformations*".

Basic idea in time-series case: map segments $\boldsymbol{X}_t = (X_t, X_{t+1}, \ldots, X_{t+m-1})$ of length m from continuously distrib., real-valued process $(X_t)_{t\in\mathbb{Z}=\{...,-1,0,1,...\}}$ onto permutations from symmetric group S_m of order m, where selected $\pi_t \in S_m = \{\pi^{[1]}, \ldots, \pi^{[m!]}\}$ expresses order among values in $\boldsymbol{X}_t$ in certain way: $(\ldots)$.

Rank representation, see Berger et al. (2019):

Entries of $\pi = (r_1, \ldots, r_m) \in S_m$ express

ranks within $\boldsymbol{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$, i. e.,

$$r_k < r_l \qquad \Leftrightarrow \qquad x_k < x_l \quad \text{or} \quad (x_k = x_l \text{ and } k < l)$$

for all $k, l \in \{1, \ldots, m\}$. Here, "$x_k = x_l$" if ties within $\boldsymbol{x}$.

Example: $(1.2, -0.7, 3.4, 1.9) \quad \mapsto \quad (2, 1, 4, 3)$,

$(1.2, -0.7, 3.4, -0.7) \quad \mapsto \quad (3, 1, 4, 2)$.

Marginal distribution of OP series (π_t) provides insights into **serial dependence structure** of original process (X_t).

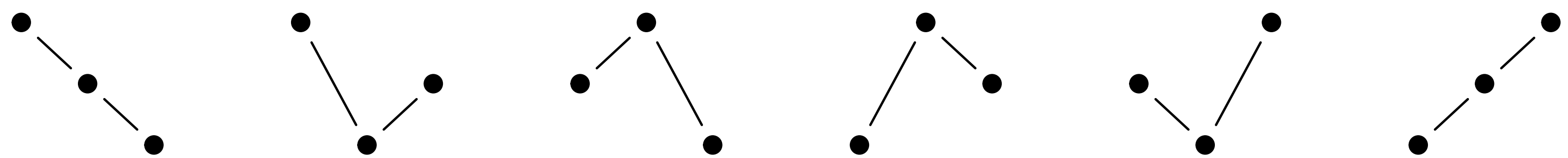
Focus on $m!$-dimensional **pmf vector** $\boldsymbol{p}$ (or sample pmf $\widehat{\boldsymbol{p}}$), with kth component being $p_k = P(\pi_t = \pi^{[k]})$.

Here, order m of OPs (thus dimension $m!$) chosen by user.

However, range of π_t quickly increases with m as $|S_m| = m!$, so estimation $\widehat{\boldsymbol{p}}$ of $\boldsymbol{p}$ quickly difficult in practice.

Therefore, in time series analysis, **convenient choice** is $m = 3$ (Bandt, 2019):

$(3,2,1)$ $(3,1,2)$ $(2,3,1)$ $(1,3,2)$ $(2,1,3)$ $(1,2,3)$



Let (X_t) be *continuously distributed* real-valued process, independent and identically distributed (i. i. d.) *under null*. Probability of ties $= 0$, so ties at most rarely in data.

Following **properties** crucial for dependence tests:

1. OPs invariant w.r.t. strictly monotonically increasing transformations of X_t. Thus, OPs do not depend on actual marginal distribution of $(X_t)_{\mathbb{N}}$ (**distribution-free** approach).
2. $(X_t)_{\mathbb{N}}$ is i. i. d. under null ($\rightarrow$ exchangeability).
   Thus, π_t discrete uniform on S_m, i. e., $P\big(\pi_t = \pi\big) = 1/m!$ for each $\pi \in S_m$ (**no parameter estimation** required).

OP-test statistics built upon $\widehat{\boldsymbol{p}}$,

closed-form asymptotics in Weiß (2022),

allow for non-parametric testing of i. i. d.-null.

Here: no time series data, but

spatial data in regular two-dimensional grid,

generated by **random field** (being i. i. d. under null).

Outline: *spatial* OPs and (refined) "types",

possible test statistics and asymptotics,

performance analyses by simulations,

real-data applications.

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Testing for Spatial Dependence in Random Fields

Asymptotics & Types

Real-valued and continuously distributed **spatial data**
occurring in regular two-dimensional grid: $(X_t)_{t\in\mathbb{Z}^2}$.
(random field, spatial process in plane, regular lattice structure)

Data rectangles $(x_t) = (x_{t_1,t_2})$ with $0 \leq t_1 \leq m$ and $0 \leq t_2 \leq n$.

Infer dependence in (X_t) via **spatial OPs** (SOPs),
due to Ribeiro et al. (2012) and Bandt & Wittfeld (2023).

$m_1 \times m_2$-SOP computed from $m_1 \times m_2$-rectangle from (x_t):

1. concatenate *rows* into vector of length $m_1 \cdot m_2$,
2. compute corresponding $(m_1 \cdot m_2)$th-order OP from $S_{m_1 \cdot m_2}$,
3. transform back into $m_1 \times m_2$-matrix in *row-wise* manner.

As $|S_{m_1 \cdot m_2}| = (m_1 \cdot m_2)!$ quickly unfeasibly large,

Bandt & Wittfeld (2023) recommend focus on 2×2-**SOPs:**

$$\mathbf{X}_t = \begin{pmatrix} X_{t_1-1,t_2-1} & X_{t_1-1,t_2} \\ X_{t_1,t_2-1} & X_{t_1,t_2} \end{pmatrix} =: \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix},$$

where $(r_1, r_2, r_3, r_4) \in S_4$ is OP of (x_1, x_2, x_3, x_4).

Bandt & Wittfeld (2023): further partition into **types**,

$$\mathcal{S}_1 = \left\{ \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 4 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 4 \\ 1 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 1 \\ 4 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 4 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 2 \\ 3 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 3 \\ 2 & 1 \end{smallmatrix}\right) \right\},$$

$$\mathcal{S}_2 = \left\{ \left(\begin{smallmatrix} 1 & 2 \\ 4 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 4 \\ 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 3 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 3 \\ 1 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 2 \\ 4 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 4 \\ 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 1 \\ 3 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 3 \\ 1 & 2 \end{smallmatrix}\right) \right\},$$

$$\mathcal{S}_3 = \left\{ \left(\begin{smallmatrix} 1 & 3 \\ 4 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 4 \\ 3 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 3 \\ 4 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 4 \\ 3 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 1 \\ 2 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 2 \\ 1 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 1 \\ 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 4 & 2 \\ 1 & 3 \end{smallmatrix}\right) \right\}.$$

Visual representation of types

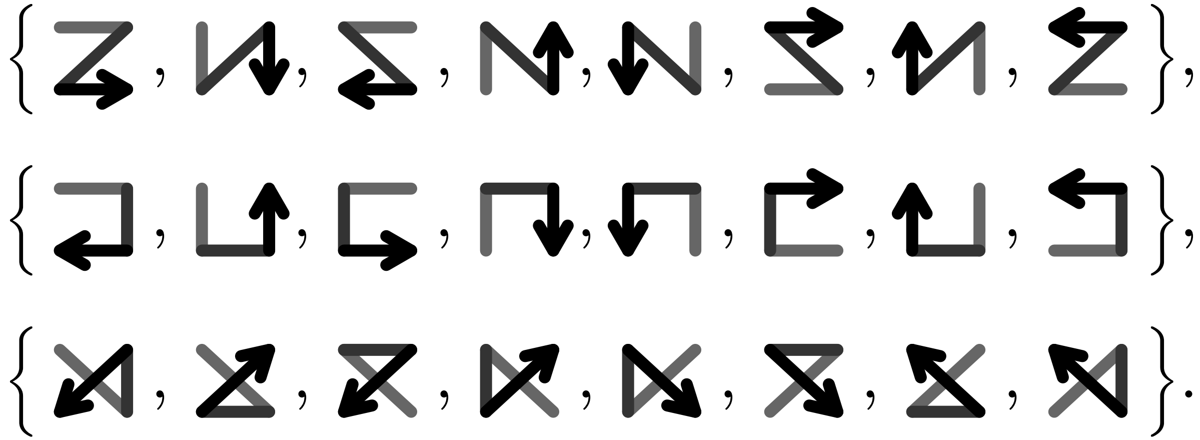
$$\mathcal{S}_1 = \left\{ \begin{pmatrix}1&2\\3&4\end{pmatrix}, \begin{pmatrix}1&3\\2&4\end{pmatrix}, \begin{pmatrix}2&1\\4&3\end{pmatrix}, \begin{pmatrix}2&4\\1&3\end{pmatrix}, \begin{pmatrix}3&1\\4&2\end{pmatrix}, \begin{pmatrix}3&4\\1&2\end{pmatrix}, \begin{pmatrix}4&2\\3&1\end{pmatrix}, \begin{pmatrix}4&3\\2&1\end{pmatrix} \right\},$$

$$\mathcal{S}_2 = \left\{ \begin{pmatrix}1&2\\4&3\end{pmatrix}, \begin{pmatrix}1&4\\2&3\end{pmatrix}, \begin{pmatrix}2&1\\3&4\end{pmatrix}, \begin{pmatrix}2&3\\1&4\end{pmatrix}, \begin{pmatrix}3&2\\4&1\end{pmatrix}, \begin{pmatrix}3&4\\2&1\end{pmatrix}, \begin{pmatrix}4&1\\3&2\end{pmatrix}, \begin{pmatrix}4&3\\1&2\end{pmatrix} \right\},$$

$$\mathcal{S}_3 = \left\{ \begin{pmatrix}1&3\\4&2\end{pmatrix}, \begin{pmatrix}1&4\\3&2\end{pmatrix}, \begin{pmatrix}2&3\\4&1\end{pmatrix}, \begin{pmatrix}2&4\\3&1\end{pmatrix}, \begin{pmatrix}3&1\\2&4\end{pmatrix}, \begin{pmatrix}3&2\\1&4\end{pmatrix}, \begin{pmatrix}4&1\\2&3\end{pmatrix}, \begin{pmatrix}4&2\\1&3\end{pmatrix} \right\}$$

by arrows along increasing rank:

$\mathcal{S}_1 =$, $\mathcal{S}_2 =$, $\mathcal{S}_3 =$



("Z-type")

("U-type")

("X-type")

Spatial Dependence: Asymptotics & Types

3 types more feasible in small data than 24 SOPs.

Type 1: monotonic behaviour both along rows and columns.

Type 2: uniquely increase/decrease only along one direction.

Type 3: lowest and highest ranks on either main or antidiagonal.

type = rank number which shares diagonal with rank 4.

Example: arthropods data from R-package `agridat`:

$$
\begin{array}{cccll}
35 & 24 & 18 & \cdots & t_2 \\
18 & 32 & 14 & \cdots & \\
\boxed{\begin{array}{cc} 17 & 21 \\ 17 & 34 \end{array}} & & \begin{array}{c} 40 \\ 25 \end{array} & \begin{array}{c} \cdots \\ \cdots \end{array} & \\
\vdots & \vdots & \vdots & \ddots & \\
t_1 & & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{cccl}
\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} & \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} & \cdots & t_2 \\
\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} & \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} & \cdots & \\
\boxed{\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}} & \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} & \cdots & \\
\vdots & \vdots & \ddots & \\
t_1 & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{cccl}
3 & 2 & \cdots & t_2 \\
1 & 3 & \cdots & \\
\boxed{1} & 3 & \cdots & \\
\vdots & \vdots & \ddots & \\
t_1 & & &
\end{array}
$$

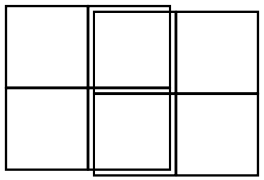
$\boldsymbol{p}, \widehat{\boldsymbol{p}}$ use **lexicographic ordering** of SOPs; $\boldsymbol{p}_0 = (\frac{1}{24}, \ldots, \frac{1}{24})^\top$.

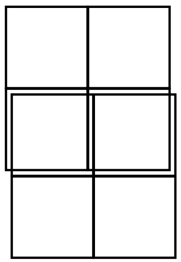
Theorem: (Weiß & Kim, 2024a)

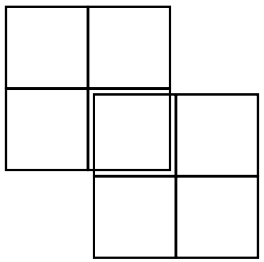
Under i. i. d.-null, $\sqrt{mn}\,\big(\widehat{\boldsymbol{p}} - \boldsymbol{p}_0\big) \xrightarrow{\text{d}} \text{N}\big(\mathbf{0}, \boldsymbol{\Sigma}_{\text{full}}\big)$

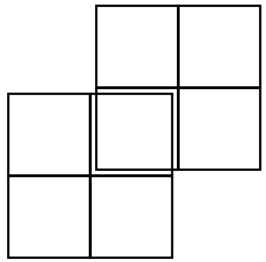
$$\boldsymbol{\Sigma}_{\text{full}} = \text{diag}(\boldsymbol{p}_0) - \boldsymbol{p}_0\,\boldsymbol{p}_0^\top + \Big(\mathbf{D} + \mathbf{D}^\top + \mathbf{A} + \mathbf{A}^\top - 4\,\boldsymbol{p}_0\,\boldsymbol{p}_0^\top\Big)$$
$$+ \Big(\mathbf{H} + \mathbf{H}^\top - 2\,\boldsymbol{p}_0\,\boldsymbol{p}_0^\top\Big) + \Big(\mathbf{V} + \mathbf{V}^\top - 2\,\boldsymbol{p}_0\,\boldsymbol{p}_0^\top\Big).$$

Considers different overlaps of 2×2-SOPs:

H: 

V: 

D: 

A: 

Closed-form expression for Σ_{full} from Weiß & Kim (2024a).

Entries of matrix $20\,160 \cdot \Sigma_{\mathrm{full}}$:

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}	π_{16}	π_{17}	π_{18}	π_{19}	π_{20}	π_{21}	π_{22}	π_{23}	π_{24}
π_1	773	−75	−99	−95	−67	−111	−75	−27	−67	41	−35	−3	−95	−35	−111	−59	181	61	41	−59	−3	−83	61	−59
π_2	−75	645	−67	−175	13	−87	101	−75	−139	−87	−59	29	13	−67	−11	13	61	−59	−11	−139	−7	−59	181	61
π_3	−99	−67	773	−111	−75	−95	−67	−35	−75	−3	−27	41	−111	181	−95	61	−35	−59	−3	61	41	−59	−59	−83
π_4	−95	−175	−111	685	−87	−107	13	−95	−11	−107	41	−59	157	−111	37	−87	−3	29	37	−11	29	41	−7	−3
π_5	−67	13	−75	−87	645	−175	−139	−59	101	29	−75	−87	−11	61	13	−59	−67	13	−7	181	−11	61	−139	−59
π_6	−111	−87	−95	−107	−175	685	−11	41	13	−59	−95	−107	37	−3	157	29	−111	−87	29	−7	37	−3	−11	41
π_7	−75	101	−67	13	−139	−11	645	−75	13	−11	−59	−7	−175	−67	−87	−139	61	181	−87	13	29	−59	−59	61
π_8	−27	−75	−35	−95	−59	41	−75	773	−59	−111	−83	−3	−95	−99	41	−67	−59	61	−111	−67	−3	−35	61	181
π_9	−67	−139	−75	−11	101	13	13	−59	645	−7	−75	−11	−87	61	−175	181	−67	−139	29	−59	−87	61	13	−59
π_{10}	41	−87	−3	−107	29	−59	−11	−111	−7	685	−3	−107	37	−95	29	−175	41	−87	157	13	37	−95	−11	−111
π_{11}	−35	−59	−27	41	−75	−95	−59	−83	−75	−3	773	−111	41	−59	−95	61	−99	−67	−3	61	−111	181	−67	−35
π_{12}	−3	29	41	−59	−87	−107	−7	−3	−11	−107	−111	685	29	41	37	−87	−95	−175	37	−11	157	−111	13	−95
π_{13}	−95	13	−111	157	−11	37	−175	−95	−87	37	41	29	685	−111	−107	−11	−3	−7	−107	−87	−59	41	29	−3
π_{14}	−35	−67	181	−111	61	−3	−67	−99	61	−95	−59	41	−111	773	−3	−75	−83	−59	−95	−75	41	−27	−59	−35
π_{15}	−111	−11	−95	37	13	157	−87	41	−175	29	−95	37	−107	−3	685	−7	−111	−11	−59	29	−107	−3	−87	41
π_{16}	−59	13	61	−87	−59	29	−139	−67	181	−175	61	−87	−11	−75	−7	645	−59	13	13	101	−11	−75	−139	−67
π_{17}	181	61	−35	−3	−67	−111	61	−59	−67	41	−99	−95	−3	−83	−111	−59	773	−75	41	−59	−95	−35	−75	−27
π_{18}	61	−59	−59	29	13	−87	181	61	−139	−87	−67	−175	−7	−59	−11	13	−75	645	−11	−139	13	−67	101	−75
π_{19}	41	−11	−3	37	−7	29	−87	−111	29	157	−3	37	−107	−95	−59	13	41	−11	685	−175	−107	−95	−87	−111
π_{20}	−59	−139	61	−11	181	−7	13	−67	−59	13	61	−11	−87	−75	29	101	−59	−139	−175	645	−87	−75	13	−67
π_{21}	−3	−7	41	29	−11	37	29	−3	−87	37	−111	157	−59	41	−107	−11	−95	13	−107	−87	685	−111	−175	−95
π_{22}	−83	−59	−59	41	61	−3	−59	−35	61	−95	181	−111	41	−27	−3	−75	−35	−67	−95	−75	−111	773	−67	−99
π_{23}	61	181	−59	−7	−139	−11	−59	61	13	−11	−67	13	29	−59	−87	−139	−75	101	−87	13	−175	−67	645	−75
π_{24}	−59	61	−83	−3	−59	41	61	181	−59	−111	−35	−95	−3	−35	41	−67	−27	−75	−111	−67	−95	−99	−75	773

By above Theorem, derive asymptotics of any aggregation.

Concerning **types**, define transformation matrix $\mathbf{T}_{\text{types}} \in \{0,1\}^{3\times 24}$ as 1 iff j^{th} SOP of type i, and 0 otherwise.

Corollary: (Bandt & Wittfeld, 2023)

Under i. i. d.-null, asymptotic covariance of type frequencies

$$\boldsymbol{\Sigma}_{\text{types}} = \mathbf{T}_{\text{types}}\,\boldsymbol{\Sigma}_{\text{full}}\,\mathbf{T}_{\text{types}}^{\top} = \frac{1}{45}\begin{pmatrix} 11 & -5 & -6 \\ -5 & 11 & -6 \\ -6 & -6 & 12 \end{pmatrix},$$

with non-zero eigenvalues $\frac{2}{5}$ and $\frac{16}{45}$.

Bandt & Wittfeld (2023) proposed following type statistics:

$$\widehat{\tau} \;=\; \widehat{p}_1 - 1/3 \quad \text{and} \quad \widehat{\kappa} \;=\; \widehat{p}_2 - \widehat{p}_3,$$
$$\widetilde{\tau} \;=\; \widehat{p}_3 - 1/3 \quad \text{and} \quad \widetilde{\kappa} \;=\; \widehat{p}_1 - \widehat{p}_2.$$

Corollary: Under i. i. d.-null,

$\sqrt{mn}\,\big(\widehat{\tau},\widehat{\kappa}\big) \stackrel{\mathrm{d}}{\to} \mathrm{N}\big(\mathbf{0},\boldsymbol{\Sigma}'\big)$ and $\sqrt{mn}\,\big(\widetilde{\tau},\widetilde{\kappa}\big) \stackrel{\mathrm{d}}{\to} \mathrm{N}\big(\mathbf{0},\boldsymbol{\Sigma}''\big)$, where

$$\boldsymbol{\Sigma}' \;=\; \frac{1}{45}\begin{pmatrix} 11 & 1 \\ 1 & 35 \end{pmatrix}, \qquad \boldsymbol{\Sigma}'' \;=\; \frac{4}{45}\begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}.$$

$\Rightarrow$ four **non-parametric tests for spatial dependence:**

$\widehat{\tau}$-test, $\widehat{\kappa}$-test, $\widetilde{\tau}$-test, $\widetilde{\kappa}$-test.

Performance analyses in Weiß & Kim (2024a):

While spatial ACF superior for *linear unilateral* DGPs, SOP-based tests often superior in presence of *outliers*, for *non-linear* DGPs, and for *bilateral* spatial DGPs.

$\tilde{\tau}$-test outperforms all other type-tests!

However, types imply a three-part partition $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Reducing real-valued spatial data set to three types only, may lose information for uncovering spatial dependence.

$\Rightarrow$ Develop "**refined types**" to preserve more information!

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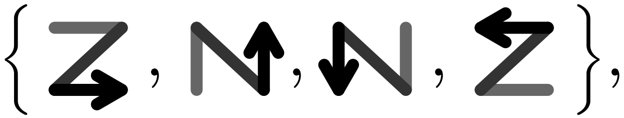
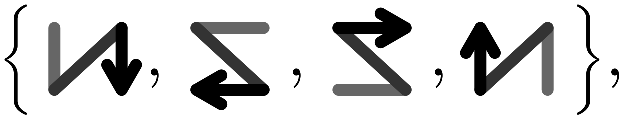
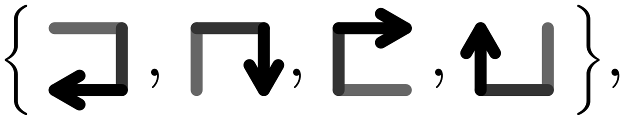
Refined Types & Entropy

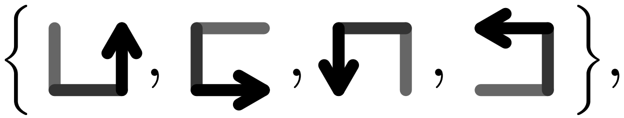
1st refinement $\mathcal{S}_i = \mathcal{S}_{i.1} \cup \mathcal{S}_{i.2}$ in Weiß & Kim (2024b):

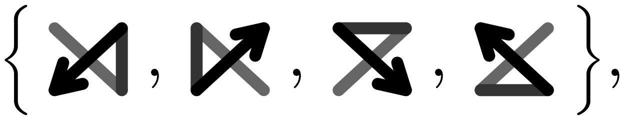
rotation types (invariant w.r.t. rotation)

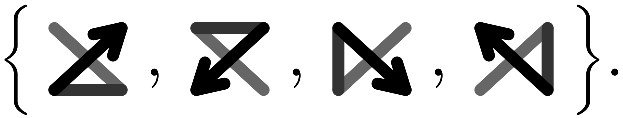
$\mathcal{S}_{1.1} = \{\pi_1, \pi_{11}, \pi_{14}, \pi_{24}\} =$ ,

$\mathcal{S}_{1.2} = \{\pi_3, \pi_8, \pi_{17}, \pi_{22}\} =$ ,

$\mathcal{S}_{2.1} = \{\pi_2, \pi_9, \pi_{18}, \pi_{20}\} =$ ,

$\mathcal{S}_{2.2} = \{\pi_5, \pi_7, \pi_{16}, \pi_{23}\} =$ ,

$\mathcal{S}_{3.1} = \{\pi_4, \pi_{12}, \pi_{15}, \pi_{19}\} =$ ,

$\mathcal{S}_{3.2} = \{\pi_6, \pi_{10}, \pi_{13}, \pi_{21}\} =$ .

1st refinement $\mathcal{S}_i = \mathcal{S}_{i.1} \cup \mathcal{S}_{i.2}$ in Weiß & Kim (2024b): **rotation types**.

Corollary: Under i. i. d.-null,

asymptotic covariance of rotation-type frequencies

$$\boldsymbol{\Sigma}_{\text{rot}} = \mathbf{T}_{\text{rot}}\,\boldsymbol{\Sigma}_{\text{full}}\,\mathbf{T}_{\text{rot}}^{\top} = \frac{1}{180}\begin{pmatrix} 23 & -1 & -5 & -5 & -6 & -6 \\ -1 & 23 & -5 & -5 & -6 & -6 \\ -5 & -5 & 11 & 11 & -6 & -6 \\ -5 & -5 & 11 & 11 & -6 & -6 \\ -6 & -6 & -6 & -6 & 25 & -1 \\ -6 & -6 & -6 & -6 & -1 & 25 \end{pmatrix},$$

with non-zero eigenvalues $\frac{1}{5}$, $\frac{8}{45}$, $\frac{13}{90}$, and $\frac{2}{15}$.

2nd refinement $\mathcal{S}_i = \mathcal{S}_{i.1} \cup \mathcal{S}_{i.2}$ in Weiß & Kim (2024b):

direction types (w.r.t. positions of maximal ranks)

$$
\begin{aligned}
\mathcal{S}_{1.1} &= \{\pi_1, \pi_8, \pi_{17}, \pi_{24}\} &&= \Big\{ \begin{pmatrix} 1 & 2 \\ \mathbf{3} & \mathbf{4} \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ \mathbf{4} & \mathbf{3} \end{pmatrix}, \begin{pmatrix} \mathbf{3} & \mathbf{4} \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} \mathbf{4} & \mathbf{3} \\ 2 & 1 \end{pmatrix} \Big\}, \\
\mathcal{S}_{1.2} &= \{\pi_3, \pi_{11}, \pi_{14}, \pi_{22}\} &&= \Big\{ \begin{pmatrix} 1 & \mathbf{3} \\ 2 & \mathbf{4} \end{pmatrix}, \begin{pmatrix} 2 & \mathbf{4} \\ 1 & \mathbf{3} \end{pmatrix}, \begin{pmatrix} \mathbf{3} & 1 \\ \mathbf{4} & 2 \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 2 \\ \mathbf{3} & 1 \end{pmatrix} \Big\}, \\
\mathcal{S}_{2.1} &= \{\pi_2, \pi_7, \pi_{18}, \pi_{23}\} &&= \Big\{ \begin{pmatrix} 1 & 2 \\ \mathbf{4} & \mathbf{3} \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ \mathbf{3} & \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{3} & \mathbf{4} \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} \mathbf{4} & \mathbf{3} \\ 1 & 2 \end{pmatrix} \Big\}, \\
\mathcal{S}_{2.2} &= \{\pi_5, \pi_9, \pi_{16}, \pi_{20}\} &&= \Big\{ \begin{pmatrix} 1 & \mathbf{4} \\ 2 & \mathbf{3} \end{pmatrix}, \begin{pmatrix} 2 & \mathbf{3} \\ 1 & \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{3} & 2 \\ \mathbf{4} & 1 \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 1 \\ \mathbf{3} & 2 \end{pmatrix} \Big\}, \\
\mathcal{S}_{3.1} &= \{\pi_4, \pi_6, \pi_{10}, \pi_{12}\} &&= \Big\{ \begin{pmatrix} 1 & \mathbf{3} \\ \mathbf{4} & 2 \end{pmatrix}, \begin{pmatrix} 1 & \mathbf{4} \\ \mathbf{3} & 2 \end{pmatrix}, \begin{pmatrix} 2 & \mathbf{3} \\ \mathbf{4} & 1 \end{pmatrix}, \begin{pmatrix} 2 & \mathbf{4} \\ \mathbf{3} & 1 \end{pmatrix} \Big\}, \\
\mathcal{S}_{3.2} &= \{\pi_{13}, \pi_{15}, \pi_{19}, \pi_{21}\} &&= \Big\{ \begin{pmatrix} \mathbf{3} & 1 \\ 2 & \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{3} & 2 \\ 1 & \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 1 \\ 2 & \mathbf{3} \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 2 \\ 1 & \mathbf{3} \end{pmatrix} \Big\}.
\end{aligned}
$$

2nd refinement $\mathcal{S}_i = \mathcal{S}_{i.1} \cup \mathcal{S}_{i.2}$ in Weiß & Kim (2024b): **direction types**.

Corollary: Under i. i. d.-null,

asymptotic covariance of direction-type frequencies

$$\boldsymbol{\Sigma}_{\mathrm{dir}} = \mathbf{T}_{\mathrm{dir}}\,\boldsymbol{\Sigma}_{\mathrm{full}}\,\mathbf{T}_{\mathrm{dir}}^{\top} = \frac{1}{1260}\begin{pmatrix} 217 & -63 & -7 & -63 & -42 & -42 \\ -63 & 217 & -63 & -7 & -42 & -42 \\ -7 & -63 & 217 & -63 & -42 & -42 \\ -63 & -7 & -63 & 217 & -42 & -42 \\ -42 & -42 & -42 & -42 & 103 & 65 \\ -42 & -42 & -42 & -42 & 65 & 103 \end{pmatrix},$$

with non-zero eigenvalues $\frac{4}{15}$, $\frac{1}{5}$, $\frac{8}{45}$ (twice), and $\frac{19}{630}$.

3rd refinement $\mathcal{S}_i = \mathcal{S}_{i.1} \cup \mathcal{S}_{i.2}$ in Weiß & Kim (2024b):

diagonal types (w.r.t. diagonal with rank 4)

$$\mathcal{S}_{1.1} = \{\pi_1, \pi_3, \pi_{22}, \pi_{24}\} = \left\{ \begin{pmatrix} \mathbf{1} & 2 \\ 3 & \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{1} & 3 \\ 2 & \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 2 \\ 3 & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 3 \\ 2 & \mathbf{1} \end{pmatrix} \right\},$$

$$\mathcal{S}_{1.2} = \{\pi_8, \pi_{11}, \pi_{14}, \pi_{17}\} = \left\{ \begin{pmatrix} 2 & \mathbf{1} \\ \mathbf{4} & 3 \end{pmatrix}, \begin{pmatrix} 2 & \mathbf{4} \\ \mathbf{1} & 3 \end{pmatrix}, \begin{pmatrix} 3 & \mathbf{1} \\ \mathbf{4} & 2 \end{pmatrix}, \begin{pmatrix} 3 & \mathbf{4} \\ \mathbf{1} & 2 \end{pmatrix} \right\},$$

$$\mathcal{S}_{2.1} = \{\pi_7, \pi_9, \pi_{20}, \pi_{23}\} = \left\{ \begin{pmatrix} \mathbf{2} & 1 \\ 3 & \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{2} & 3 \\ 1 & \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 1 \\ 3 & \mathbf{2} \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 3 \\ 1 & \mathbf{2} \end{pmatrix} \right\},$$

$$\mathcal{S}_{2.2} = \{\pi_2, \pi_5, \pi_{16}, \pi_{18}\} = \left\{ \begin{pmatrix} 1 & \mathbf{2} \\ \mathbf{4} & 3 \end{pmatrix}, \begin{pmatrix} 1 & \mathbf{4} \\ \mathbf{2} & 3 \end{pmatrix}, \begin{pmatrix} 3 & \mathbf{2} \\ \mathbf{4} & 1 \end{pmatrix}, \begin{pmatrix} 3 & \mathbf{4} \\ \mathbf{2} & 1 \end{pmatrix} \right\},$$

$$\mathcal{S}_{3.1} = \{\pi_{13}, \pi_{15}, \pi_{19}, \pi_{21}\} = \left\{ \begin{pmatrix} \mathbf{3} & 1 \\ 2 & \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{3} & 2 \\ 1 & \mathbf{4} \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 1 \\ 2 & \mathbf{3} \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 2 \\ 1 & \mathbf{3} \end{pmatrix} \right\},$$

$$\mathcal{S}_{3.2} = \{\pi_4, \pi_6, \pi_{10}, \pi_{12}\} = \left\{ \begin{pmatrix} 1 & \mathbf{3} \\ \mathbf{4} & 2 \end{pmatrix}, \begin{pmatrix} 1 & \mathbf{4} \\ \mathbf{3} & 2 \end{pmatrix}, \begin{pmatrix} 2 & \mathbf{3} \\ \mathbf{4} & 1 \end{pmatrix}, \begin{pmatrix} 2 & \mathbf{4} \\ \mathbf{3} & 1 \end{pmatrix} \right\},$$

3rd refinement $\mathcal{S}_i = \mathcal{S}_{i.1} \cup \mathcal{S}_{i.2}$ in Weiß & Kim (2024b):

diagonal types.

Corollary: Under i. i. d.-null,

asymptotic covariance of diagonal-type frequencies

$$\Sigma_{\text{diag}} = \ldots = \frac{1}{1260}\begin{pmatrix} 133 & 21 & -35 & -35 & -42 & -42 \\ 21 & 133 & -35 & -35 & -42 & -42 \\ -35 & -35 & 153 & 1 & -80 & -4 \\ -35 & -35 & 1 & 153 & -4 & -80 \\ -42 & -42 & -80 & -4 & 103 & 65 \\ -42 & -42 & -4 & -80 & 65 & 103 \end{pmatrix},$$

with non-zero eigenvalues $\frac{1}{5}$, $\frac{8}{45}$, $\frac{19}{126}$, and $\frac{4}{45}$.

Test statistics: Like in Bandt (2019); Weiß (2022), we use entropy, extropy, and distance to white noise,

$$\widehat{H} := H(\widehat{\boldsymbol{p}}) \;=\; -\textstyle\sum_{k=1}^{q} \widehat{p}_k \ln \widehat{p}_k,$$

$$\widehat{H}_{\text{ex}} := H_{\text{ex}}(\widehat{\boldsymbol{p}}) \;=\; -\textstyle\sum_{k=1}^{q} \big(1-\widehat{p}_k\big) \ln\big(1-\widehat{p}_k\big),$$

$$\widehat{\Delta} := \Delta(\widehat{\boldsymbol{p}}) \;=\; \textstyle\sum_{k=1}^{q} \big(\widehat{p}_k - 1/q\big)^2.$$

Adapting **Theorem 2.1** in Weiß (2022),

$$mn\,\widehat{\Delta}, \quad -mn\,\tfrac{2}{q}\Big(\widehat{H} - \ln q\Big), \quad -2mn\,(1-\tfrac{1}{q})\,\Big(\widehat{H}_{\text{ex}} - (q-1)\,\ln\big(\tfrac{q}{q-1}\big)\Big)$$

asymptotically distributed like $\sum_{i=1}^{l} \lambda_i \cdot \chi^2_{r_i}$,

where $\lambda_1, \ldots, \lambda_l$ non-zero eigenvalues of $\boldsymbol{\Sigma}$.

Corollary in Weiß & Kim (2024b): For **ordinary types**,

$$mn\,\widehat{\Delta},\quad -\tfrac{2}{3}\,mn\left(\widehat{H}-\ln 3\right),\quad -\tfrac{4}{3}\,mn\left(\widehat{H}_{\text{ex}}-2\,\ln\left(\tfrac{3}{2}\right)\right)$$

asymptotically distributed like $\frac{2}{5}\,\chi_1^2 + \frac{16}{45}\,\chi_1^2.$

For **refined types**, asymptotic distribution of

$$mn\,\widehat{\Delta},\quad -\tfrac{1}{3}\,mn\left(\widehat{H}-\ln 6\right),\quad -\tfrac{10}{6}\,mn\left(\widehat{H}_{\text{ex}}-5\,\ln\left(\tfrac{6}{5}\right)\right)$$

is $\frac{1}{5}\,\chi_1^2 + \frac{8}{45}\,\chi_1^2 + \frac{13}{90}\,\chi_1^2 + \frac{2}{15}\,\chi_1^2$ if rotation types,

$\frac{4}{15}\,\chi_1^2 + \frac{1}{5}\,\chi_1^2 + \frac{8}{45}\,\chi_2^2 + \frac{19}{630}\,\chi_1^2$ if direction types,

$\frac{1}{5}\,\chi_1^2 + \frac{8}{45}\,\chi_1^2 + \frac{19}{126}\,\chi_1^2 + \frac{4}{45}\,\chi_1^2$ if diagonal types.

Critical values for non-parametric entropy-like tests

via R package `CompQuadForm`.

Important $(1-\alpha)$-quantiles of quadratic-form distributions:

α	Ordinary	Rotation	Direction	Diagonal
0.10	1.740201	1.279915	1.637740	1.216170
0.05	2.265401	1.566739	1.999264	1.497222
0.01	3.487299	2.210104	2.813519	2.133017

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MATH STAT

Testing for Spatial Dependence in Random Fields

Empirical Investigations

Performance analyses by simulations in Weiß & Kim (2024b).

For power analyses, various unilateral and bilateral spatial DGPs:

Unilateral:

$$\begin{array}{ccc} (t_1-1,t_2-1) & & (t_1-1,t_2) \\ & \searrow & \downarrow \\ (t_1,t_2-1) & \rightarrow & (\boldsymbol{t_1,t_2}) \end{array}$$

Bishop:

$$\begin{array}{ccccc} (t_1-1,t_2-1) & & & & (t_1-1,t_2+1) \\ & \searrow & & \swarrow & \\ & & (\boldsymbol{t_1,t_2}) & & \\ & \nearrow & & \nwarrow & \\ (t_1+1,t_2-1) & & & & (t_1+1,t_2+1) \end{array}$$

Rook:

$$\begin{array}{ccccc} & & (t_1-1,t_2) & & \\ & & \downarrow & & \\ (t_1,t_2-1) & \rightarrow & (\boldsymbol{t_1,t_2}) & \leftarrow & (t_1,t_2+1) \\ & & \uparrow & & \\ & & (t_1+1,t_2) & & \end{array}$$

Brief summary:

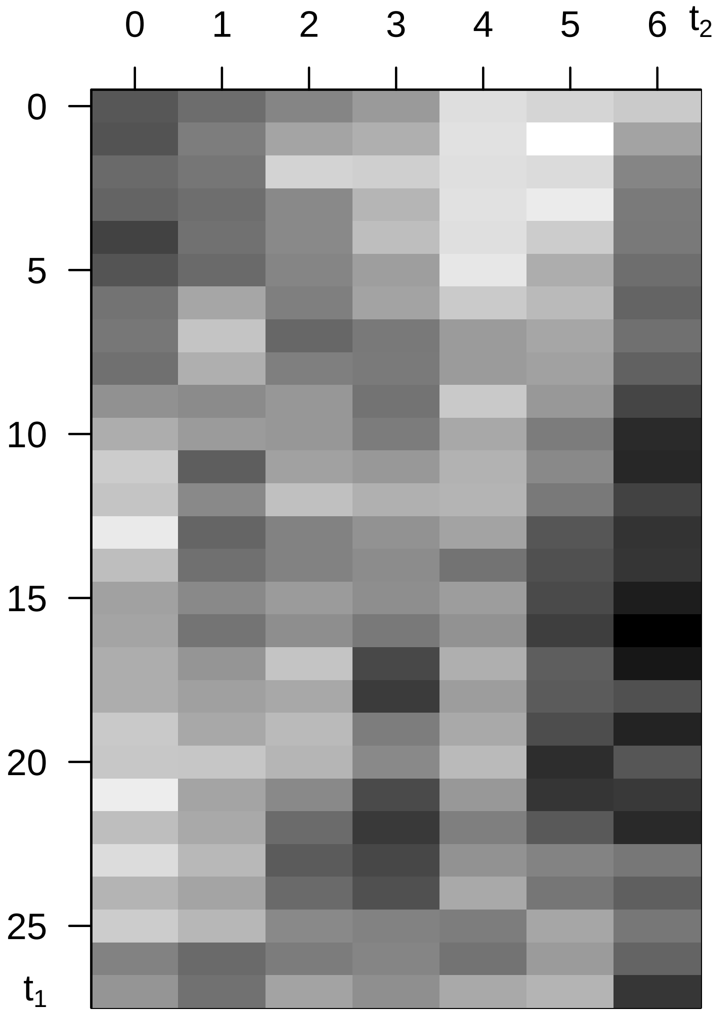
- Size: all tests hold nominal level sufficiently well.
- $\tilde{\tau}$-test excellent choice for uncovering spatial dependence, but in many cases, further improvements by appropriate form of refined type. (...)

Brief summary:

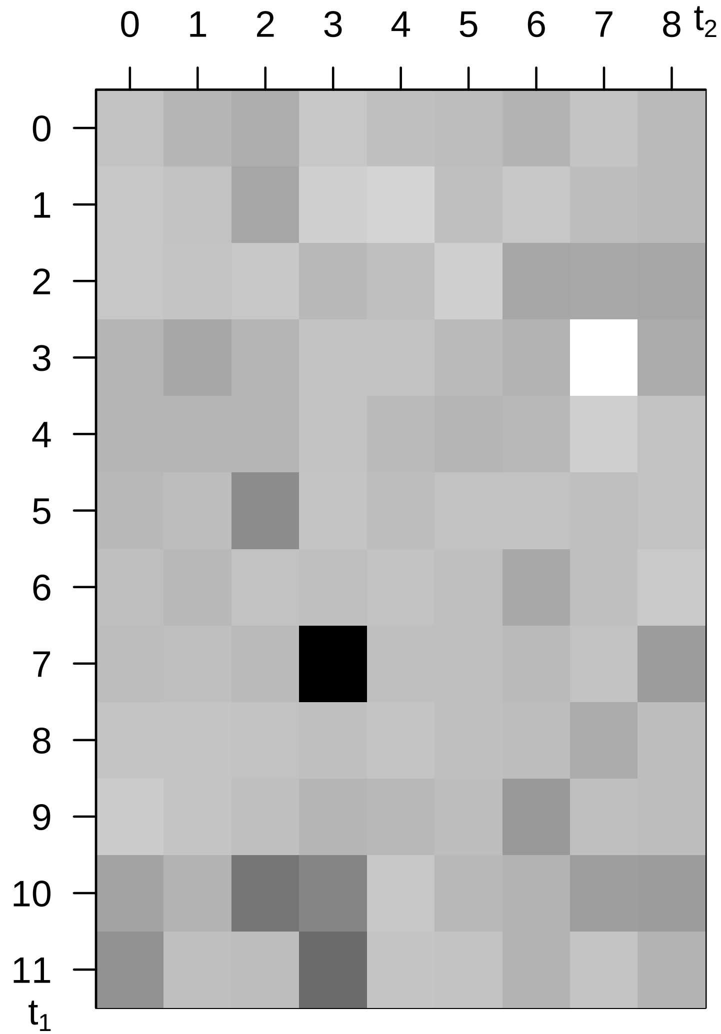
- For unilateral DGPs, diagonal types superior in most cases.
- For bilateral DGPs, $\tilde{\tau}$-test often superior.
- For bil. DGPs with Rook structure, direction types superior.
- If refined types superior, then extropy most powerful.

Weiß & Kim (2024b): two **real-world data examples**, **yield of barley** (in kg) in 28×7-grid ($m = 27, n = 6$) from uniformity trial experiment (Kempton & Howes, 1981); **population change** in Turku region between 2005 and 2022 in $1\,\text{km} \times 1\,\text{km}$ grid ($m = 11$, $n = 8$), see Statistics Finland.

Yield of barley:



Population change:



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Spatial Dependence: Empirical Investigations

MATH STAT

Yield of barley: all tests reject i. i. d.-null (P-values $\approx 10^{-9}$).

Particularly large extropy value for direction types,

mainly $\mathcal{S}_{1.2}$ (32.1%) and $\mathcal{S}_{2.2}$ (36.4%) $\Rightarrow$ "dominant columns".

$$\mathcal{S}_{1.2} = \Big\{\left(\begin{smallmatrix}1&\mathbf{3}\\2&\mathbf{4}\end{smallmatrix}\right),\left(\begin{smallmatrix}2&\mathbf{4}\\1&\mathbf{3}\end{smallmatrix}\right),\left(\begin{smallmatrix}\mathbf{3}&1\\\mathbf{4}&2\end{smallmatrix}\right),\left(\begin{smallmatrix}\mathbf{4}&2\\\mathbf{3}&1\end{smallmatrix}\right)\Big\},\qquad \mathcal{S}_{2.2} = \Big\{\left(\begin{smallmatrix}1&\mathbf{4}\\2&\mathbf{3}\end{smallmatrix}\right),\left(\begin{smallmatrix}2&\mathbf{3}\\1&\mathbf{4}\end{smallmatrix}\right),\left(\begin{smallmatrix}\mathbf{3}&2\\\mathbf{4}&1\end{smallmatrix}\right),\left(\begin{smallmatrix}\mathbf{4}&1\\\mathbf{3}&2\end{smallmatrix}\right)\Big\}.$$

Kempton & Howes (1981): *"sowing, harvesting and all intermediate farming practices were carried out column by column"*.

Population change: extremely large value at $\boldsymbol{t} = (7, 3)$,

but SOPs robust against outliers (by contrast to spatial ACF).

Only diagonal types reject (extropy's P-value 0.008), mainly

$\mathcal{S}_{1.2}$ (21.6%) and $\mathcal{S}_{2.2}$ (27.3%) $\Rightarrow$ "dominant antidiagonal".

$$\mathcal{S}_{1.2} = \Big\{\left(\begin{smallmatrix}2&\mathbf{1}\\\mathbf{4}&3\end{smallmatrix}\right),\left(\begin{smallmatrix}2&\mathbf{4}\\\mathbf{1}&3\end{smallmatrix}\right),\left(\begin{smallmatrix}3&\mathbf{1}\\\mathbf{4}&2\end{smallmatrix}\right),\left(\begin{smallmatrix}3&\mathbf{4}\\\mathbf{1}&2\end{smallmatrix}\right)\Big\},\qquad \mathcal{S}_{2.2} = \Big\{\left(\begin{smallmatrix}1&\mathbf{2}\\\mathbf{4}&3\end{smallmatrix}\right),\left(\begin{smallmatrix}1&\mathbf{4}\\\mathbf{2}&3\end{smallmatrix}\right),\left(\begin{smallmatrix}3&\mathbf{2}\\\mathbf{4}&1\end{smallmatrix}\right),\left(\begin{smallmatrix}3&\mathbf{4}\\\mathbf{2}&1\end{smallmatrix}\right)\Big\}.$$

Conclusions & Outlook

SOPs and (refined) types well-interpretable and robust.

If data continuously distributed, then non-parametric tests.

Easily implemented in practice due to closed-form asymptotics.

Direction and diagonal types advantageous for unilateral and certain bilateral DGPs, where extropy statistic most powerful.

Work in progress & future research:

- Control charts based on SOPs and (refined) types, in analogy to Weiß & Testik (2023);
- SOPs based on "generalized OPs" where ties explicitly accounted for, in analogy to Weiß & Schnurr (2024).

Thank You

for Your Interest!

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Universität der Bundeswehr Hamburg

MATH
STAT

Christian H. Weiß

Department of Mathematics & Statistics

Helmut Schmidt University, Hamburg

`weissc@hsu-hh.de`

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